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Double-diffusive convection in an inclined slot

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We examine two-dimensional motion of a stably stratified fluid containing two solutes with different molecular diffusivities in an inclined slot. The two solutes have continuous opposing gradients with the slower-diffusing one more dense at the bottom. It is found that, in the steady state, there exists a slow upward flow along the slope driven by the slight buoyancy difference near the wall, not unlike the solution found by Phillips (1970) for a single solute. The magnitude of the flow is less than that in Phillips' solution by a factor of approximately $(1-\lambda)/(1-\lambda\tau)$, where λ is the ratio of the density gradient and τ^{-1} is the ratio of the diffusivity of the faster-diffusing solute to that of the slower-diffusing one. For the time-dependent flow resulting from switching on the diffusivities at $t = 0$, there may be a dramatic reversal of the flow near the walls depending on the relative magnitude of λ and τ. If λ is somewhat greater than τ, the initial flow is downward, along the slope, reaching a maximum magnitude about one order of magnitude greater than the steady-state value. Then the 'reverse' flow gradually diminishes and approaches the steady state rather slowly. For $\lambda \lesssim \tau$, the approach to the steady state is monotonic; there is no 'reverse' flow near the wall. The existence of the downward flow, which was observed by Turner & Chen (1974), may lead to double-diffusive instabilities which eventually result in horizontal convecting layers.

1. Introduction

When a stably stratified fluid containing a single solute is in the presence of a sloping impermeable wall, the constant-density lines must curve in the neighbourhood of the wall to meet it normally. This slight displacement of the constant-density lines generates a buoyancy force in the fluid. Phillips (1970) and Wunsch (1970) have shown that under steady-state conditions, when the buoyancy forces are balanced by the viscous forces, a 'buoyancy' layer flows steadily upwards along the slope. In the case of a stably stratified fluid containing two solutes with differing molecular diffusivities, such as sugar and salt, the motion along a sloping wall has been observed to be quite different. Turner & Chen (1974) have carried out such an experiment, in which a tank was filled with continuous opposing gradients of sugar and salt with the slower-diffusing sugar more dense at the bottom. The overall density gradient of the double-diffusive system was stable. Into this tank, a plate was inserted at an angle of 45°. After the initial transients had died out, there was a flow *down* the slope immediately adjacent to the wall.

Away from this downflow layer, there was an upward flow. The upward flow brought the sugar above the salt, thus generating fingers, which eventually resulted in a series of horizontal layers separated by interfaces with sharp gradients. For this experiment, the ratio λ of density gradients was $\approx 0{\cdot}9$, where

$$\lambda = \alpha \frac{\partial T}{\partial \zeta} \Big/ \beta \frac{\partial S}{\partial \zeta}, \tag{1}$$

in which T and S denote the faster- and slower-diffusing components, respectively, ζ is the vertical co-ordinate and

$$\alpha = -\rho^{-1}\,\partial\rho/\partial T, \quad \beta = \rho^{-1}\,\partial\rho/\partial S.$$

The density of the fluid is

$$\rho = \rho_0[1-\alpha(T-T_0)+\beta(S-S_0)],$$

in which the subscript zero denotes reference values. Chen & Wong (1974) have reported some preliminary experimental results on the effect of reducing λ on the subsequent occurrence of cellular convection. As λ is reduced to about $0{\cdot}8$, the upward-flowing outer layer disappears while the slowly descending layer is still visible. Horizontal layers are still generated along the sloping wall although they are not a consequence of finger instability. When λ is reduced below $0{\cdot}6$, all layering activities cease. It must be noted here that the value $0{\cdot}6$ is for a 45° wall. As the angle of inclination varies, the critical value of λ may vary accordingly. It is further noted that the experiment was time-dependent whereas Phillips' solution is for steady-state motion.

To seek a rational explanation for such drastically different initial flow behaviour for a single- and double-diffusive system, we investigate the problem of double-diffusive convection in a inclined slot. The results show that in the steady state, for which an analytical solution is obtained, there is an upward flow along the slope. The magnitude of the velocity is reduced from that for a single-diffusive case by a factor of approximately $(1-\lambda)/(1-\lambda\tau)$, where τ is the ratio of the diffusivities κ_s/κ_t. However, it is in the time-dependent state that pronounced differences are found between the single- and double-diffusive systems. The time dependence is introduced by switching on the diffusivities at $t = 0$. For the single-diffusive case, the approach to the steady state is monotonic; that is an upward flow of small magnitude is generated initially and approaches the steady-state value in a comparatively short time. For the double-diffusive case, the flow behaviour depends on the value of λ. For values of λ somewhat smaller than τ, the flow behaves in the same way as a single-diffusive system. If λ is somewhat larger than τ, the initial flow is *down* the slope. This downward flow slowly diminishes and finally reverses itself and approaches the steady state rather slowly. It is conjectured that an initial reverse flow of sufficient magnitude is conducive to the occurrence of double-diffusive instability, not unlike the situation for lateral heating of a salinity gradient (Chen 1974), which eventually leads to horizontal cellular convection. In this paper, we focus our attention on the initial flow behaviour: the subsequent secondary flow must be analysed in terms of the stability of the time-dependent basic flow field.

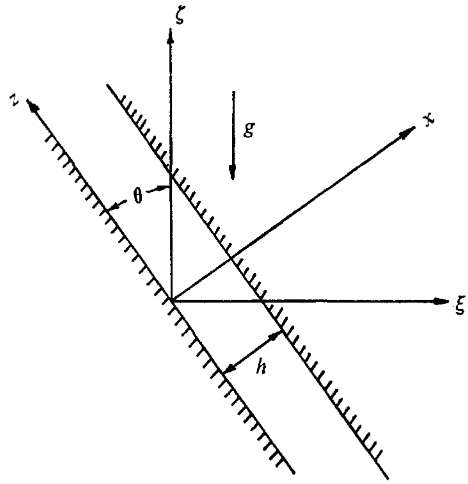

FIGURE 1. Co-ordinate system.

2. Governing equations and steady-state solution

Consider two-dimensional motion in an infinite slot of width h which is inclined with respect to the vertical axis ζ at an angle θ. Let T and S denote the faster- and slower-diffusing components, respectively, and κ_t and κ_s denote their respective molecular diffusivities. The initial T gradient $\partial T/\partial\zeta$ is positive and the initial S gradient $\partial S/\partial\zeta$ is negative with the ratio of the density gradients $\lambda < 1$, so that the fluid is statically stable. Let the z axis be along the left wall of the slot and the x axis normal to that wall (see figure 1). We seek a solution in which the motion is parallel to the z axis. We separate out the background z dependence of the T and S distributions by defining new variables T_1 and S_1:

$$T = T_0 + z\cos\theta + T_1(x,t), \quad S = S_0 - z\cos\theta + S_1(x,t), \tag{2}$$

where the subscripts zero denote quantities evaluated at $\zeta = 0$. The non-dimensional diffusion equations for the vorticity $\omega(x)$, $T_1(x)$ and $S_1(x)$ based on the Boussinesq approximation are

$$\frac{1}{\sigma}\frac{\partial\omega}{\partial t} = \frac{\partial^2\omega}{\partial x^2} + R_s\cos\theta\left[\lambda\frac{\partial T_1}{\partial x} + \frac{\partial S_1}{\partial x}\right] + R_s\cos\theta\sin\theta\,(1-\lambda), \tag{3}$$

$$\partial T_1/\partial t = \partial^2 T_1/\partial x^2 - w\cos\theta, \tag{4}$$

$$\partial S_1/\partial t = \tau\,\partial^2 S_1/\partial x^2 + w\cos\theta, \tag{5}$$

in which the S Rayleigh number $R_s = -g\beta(\partial S/\partial\zeta)\,h^4(\nu\kappa_t)^{-1}$, $\sigma = \nu/\kappa_t$ and w is the z component of the velocity. The initial and boundary conditions are

$$T_1 = -S_1 = x\sin\theta \quad \text{at} \quad t = 0, \tag{6}$$

$$w = \partial T_1/\partial x = \partial S_1/\partial x = 0 \quad \text{at} \quad x = 0, 1 \quad \text{for} \quad t > 0. \tag{7}$$

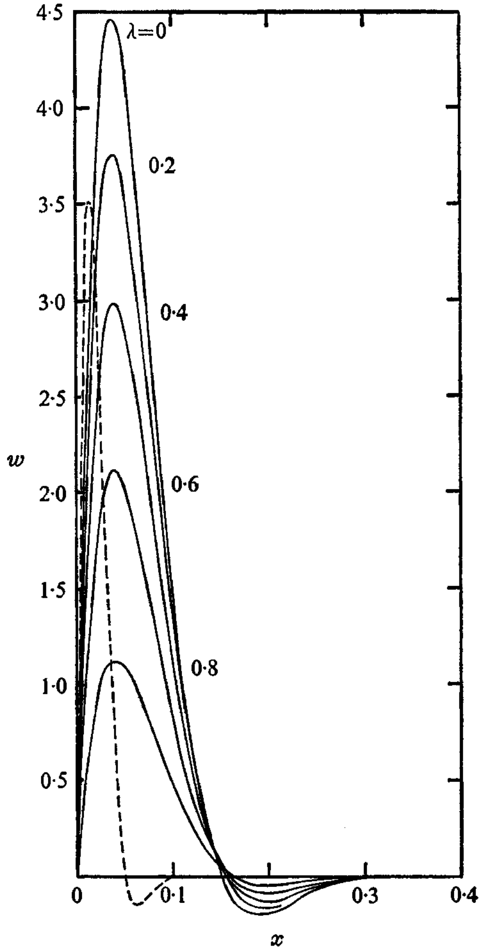

FIGURE 2. Steady-state velocity profiles for $\theta = 45°$. ——, $R_s = 5\times 10^5$, various λ; – – –, $R_s = 5\times 10^7$, $\lambda = 0{\cdot}8$.

To render the governing equations non-dimensional, we have used h as the characteristic length, h^2/κ_t as the characteristic time and $h\,\partial T/\partial\zeta$ and $-h\,\partial S/\partial\zeta$ as characteristic values of T and S, respectively.

By considering $\partial/\partial t = 0$ and replacing $\partial T_1/\partial x$ and $\partial S_1/\partial x$ by their respective forms in terms of w, we obtain for the steady state a fourth-order equation

$$d^4w/dx^4 + R_s\cos^2\theta(\tau^{-1}-\lambda)\,w = 0 \tag{8}$$

with boundary conditions

$$w = 0, \quad d^3w/dx^3 = R_s\cos\theta\sin\theta(1-\lambda) \quad \text{at} \quad x = 0, 1. \tag{9}$$

The solution is

$$w = 2FK\tan\theta\,\frac{\sin Kx\sinh K(1-x) - \sin K(1-x)\sinh Kx}{\sin K + \sinh K} \tag{10}$$

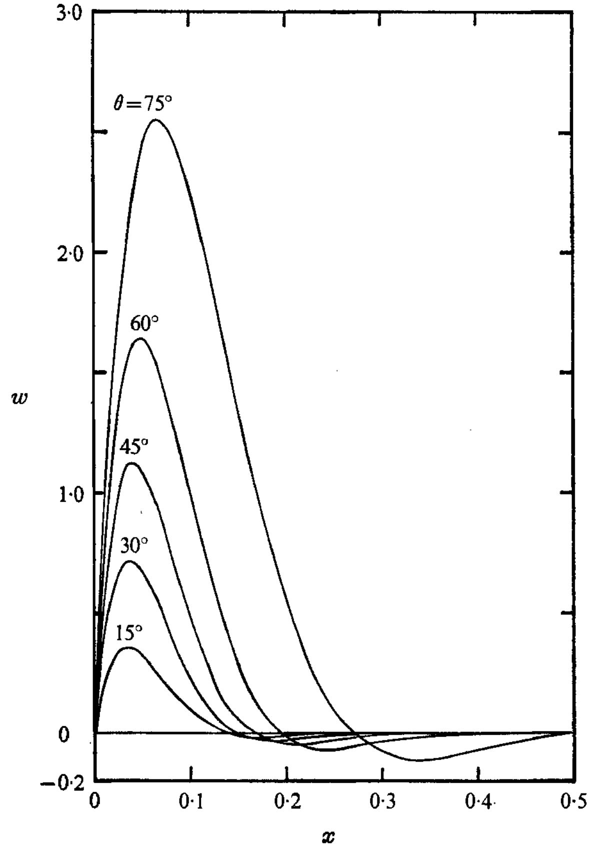

FIGURE 3. Effect of inclination angle on steady-state velocity profile for $R_s = 5\times 10^5$, $\lambda = 0{\cdot}8$.

and to within a constant

$$T_1 = -\tau S_1 = (F/K)\sin\theta\left[\frac{\cos Kx\cosh K(1-x)-\cos K(1-x)\cosh Kx}{\sin K+\sinh K}+Kx\right], \quad (11)$$

where $F = (1-\lambda)/(\tau^{-1}-\lambda), \quad K = [\tfrac{1}{4}(\tau^{-1}-\lambda)R_s\cos^2\theta]^{\frac{1}{4}}.$

It may be seen that in the steady state the solution is independent of σ. For $\lambda = 0$, the solution reduces to that given by Phillips (1970).

The steady-state velocity profiles for $\theta = 45°$, $R_s = 5\times 10^5$ and a range of values of λ from 0 to 0·8 are shown in figure 2. Owing to the antisymmetric nature of the solution, only the left half of the slot is shown. Generally, there is an upward flow along the left wall. As the adverse gradient of salt is increased, the magnitude of the velocity decreases; the reduction is nearly proportional to $(1-\lambda)/(1-\lambda\tau)$. The actual magnitude of the velocity is very small. If we take $h = 1$ cm and $\kappa_t = 10^{-5}\,\text{cm}^2\,\text{s}^{-1}$, then $w = 1$ corresponds to $10^{-5}\,\text{cm}\,\text{s}^{-1}$. For the purpose of comparison, the velocity profile for $R_s = 5\times 10^7$ and $\lambda = 0{\cdot}8$ is also shown. The effect of increasing R_s is to confine the motion closer to the wall and at the same time to increase the peak value of the velocity. We note here that the Reynolds number based on the maximum velocity and the gap width is

w/σ, which is rather small. The instability we observed in the experiments is double-diffusive rather than hydrodynamic in nature.

The effect of the angle of inclination of the sloping wall on the steady-state velocity profile is shown in figure 3. Velocity profiles for $R_s = 5 \times 10^5$, $\lambda = 0{\cdot}8$ and a range of θ from 15° to 75° are presented. It may be seen that the peak velocity as well as the lateral extent of the 'buoyancy' layer increases as the inclination angle is increased.

It is interesting to note that the convective salt transport in a narrow fissure will be reduced by the presence of a negative temperature gradient. The expression for the salt flux is the same as equation (25) of Phillips (1970) except that γd should be read as K and the entire expression multiplied by $(1-\lambda)^2/(1-\lambda\tau)^2$. Suppose that the difference in the salinity of the sea water between the bottom and the top of the fissure is 2‰ and the temperature difference between the two corresponding points is 10 °C. The value of λ is 0·6 so the salt flux would be about 80 % of the value obtained in the absence of any temperature gradient.

3. Time-dependent solution

The source terms in the vorticity transport equation (3) may be rewritten as

$$R_s \cos\{(\sin\theta + \partial S_1/\partial x) - \lambda(\sin\theta - \partial T_1/\partial x)\}. \tag{12}$$

By the nature of the problem $-\partial S_1/\partial x \leqslant \sin\theta$. Therefore the first term is always a source and the second always a sink of vorticity. Since $\partial T_1/\partial x = -\partial S_1/\partial x = \sin\theta$ initially throughout the slot, the effect of the sources and sinks is confined within the diffusion layers in the neighbourhood of the two walls. The resulting effect of these two terms in (12) depends on the relative magnitude of λ and τ, since $\sqrt{\tau}$ is the ratio of the thicknesses of diffusive layers for S and T. For $\lambda = 0$, there are only sources of vorticity near the walls. As λ increases, the source effect diminishes, and eventually the character of the initial flow will be changed because of the presence of sinks near the wall. The results of calculations given below show that the critical value of λ at which reverse flow occurs is approximately τ.

The basic equations (3)–(5) were integrated numerically using a procedure given by Chen (1974). An explicit scheme was used for the time integration. The velocity was written in terms of a stream function which is related to the vorticity by a one-dimensional Poisson equation. The latter was solved by inverting a tridiagonal matrix using the method of Gaussian elimination. In order to ensure numerical stability of the explicit integration scheme, $\Delta t < 0{\cdot}50\Delta x^2/\sigma$. For a salt–sugar solution, for which $\sigma \sim 700$ and $\tau \sim \frac{1}{3}$, the time steps were extremely small. Since initially the flow is controlled by the diffusion of T and S, and the flow becomes independent of σ in the steady state, it is expected that the general character of the flow does not depend crucially on the value of σ. We made a test run for $\sigma = 7$ and 700 with all other parameters remaining the same: $R_s = 5 \times 10^5$, $\theta = 45°$, $\lambda = 0{\cdot}8$ and $\tau = \frac{1}{3}$. The results are shown in figure 4. At the two times shown, $t = 0{\cdot}0075$ and $0{\cdot}02$, there are no drastic differences between the two solutions. It is fortuitous that at $t = 0{\cdot}02$ the two solutions are almost identical. At some later time, the results for $\sigma = 7$ will overtake those for

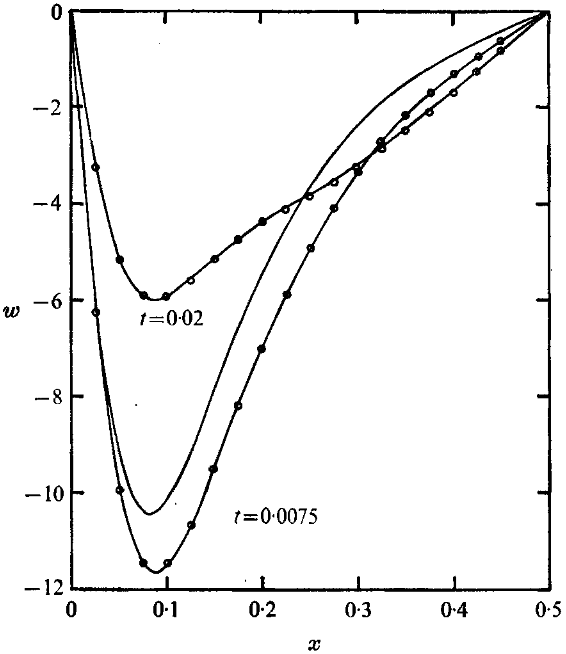

FIGURE 4. Effect of σ on time-dependent flow. $\theta = 45°$, $R_s = 5\times 10^5$. —○—, $\sigma = 7$; ——, $\sigma = 700$.

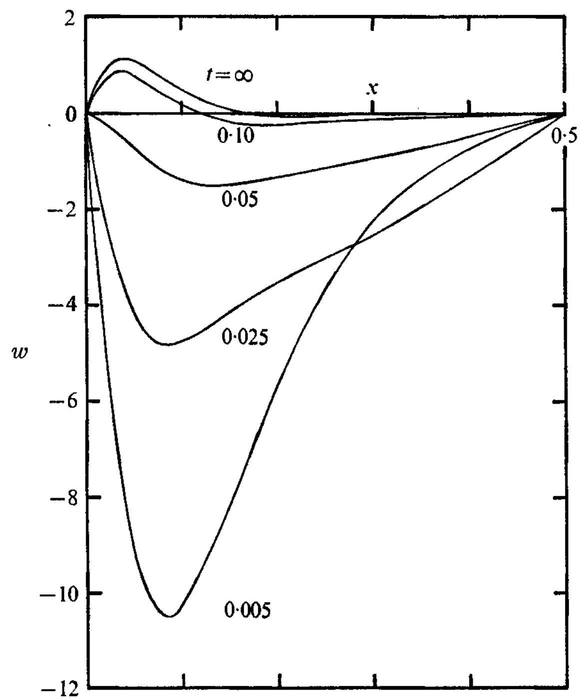

FIGURE 5. Approach to steady state. $\theta = 45°$, $R_s = 5\times 10^5$, $\lambda = 0{\cdot}8$.

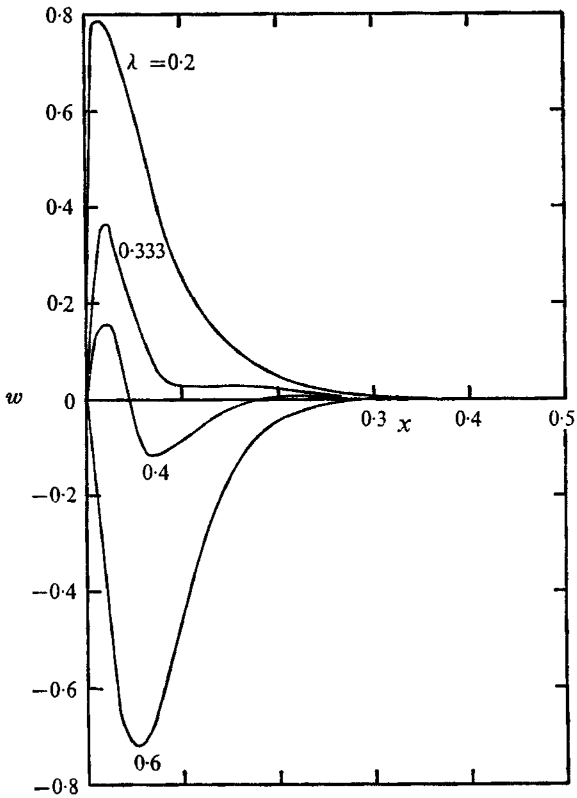

FIGURE 6. Effect of λ on the velocity profile at $t = 0{\cdot}001$. $\theta = 45°$, $R_s = 5\times 10^5$.

$\sigma = 700$. To reach $t = 0{\cdot}02$, 10 s of computing time were required for $\sigma = 7$ and 900 s for $\sigma = 700$ on an IBM 370/158 computer. In view of the fact that the value of σ is not of primary importance in determining the general character of flow and that the time saving is enormous, all subsequent calculations were made with $\sigma = 7$ and $\tau = \frac{1}{3}$. The number of grid points used was 41. By numerical experiment, it was found that there is virtually no difference between these results and those obtained using 81 or 161 grid points. The relative difference between the maximum velocities calculated using 41 and 81 grid points is about 5 % at $t = 0{\cdot}025$. The steady-state value of the maximum velocity reached by using 41 grid points is within 10 % of that calculated from (10).

For $\lambda = 0$, the approach to the steady state is monotonic. An upward flow of small magnitude is first generated near the left wall, and approaches the steady state in a comparatively short time, $t = 0{\cdot}025$ or about 40 min if $h = 1$ cm. The approach to the steady state for $\lambda = 0{\cdot}8$ is quite different. The initial flow is down the slope and reaches a maximum speed about one order of magnitude larger than the steady-state value. The downward flow gradually diminishes and finally reverses itself into an upward flow. The steady state is approached in about 4 h for $h = 1$ cm. Velocity profiles at selected times are shown in figure 5.

For the sake of clarity, profiles for $t < 0{\cdot}005$ are not shown. At these early times, the motion is confined to a region closer to the wall. For example, at $t = 0{\cdot}001$, the minimum w occurs at $x \sim 0{\cdot}05$ and the entire layer extends to only $x \sim 0{\cdot}2$.

It is anticipated that the initial flow characteristics are functions of λ and τ. Since all calculations were carried out for $\tau = \frac{1}{3}$, λ is the only parameter controlling the behaviour of the flow. Calculations have been made of the velocity profiles at a very early time of $t = 0{\cdot}001$ for $\lambda = 0{\cdot}2$, $0{\cdot}333$, $0{\cdot}4$ and $0{\cdot}6$. The results are shown in figure 6. The initial flow changes its character at $\lambda = 0{\cdot}4$. In fact for $\lambda = 0{\cdot}4$, at $t = 0{\cdot}003$, the velocity is negative over the entire left half of the slot. The results for $\theta = 30°$ and $60°$ behave in a similar manner.

As expected, for $\lambda = 0{\cdot}8$, as θ increases (decreases) the magnitude of the downward flow increases (decreases) and the time needed to reach the steady state becomes longer (shorter). It is conjectured that the reverse flow alters the T and S distributions in such a way that a suitable perturbation to the position of a fluid parcel will result in a gain or loss of buoyancy owing to the difference in the diffusivities of T and S. The most unstable disturbance will probably have a wavelength close to the thickness of the horizontal layers observed in the experiment. It is also expected that at larger values of θ the critical value of λ above which cellular convection occurs will be correspondingly smaller. All of these conjectures await further experimental and theoretical confirmation.

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REFERENCES

Chen, C. F. 1974 Onset of cellular convection in a salinity gradient due to a lateral temperature gradient. *J. Fluid Mech.* **57**, 289–306.

Chen, C. F. & Wong, S. B. 1974 Double-diffusive convection along a sloping wall. *Bull. Am. Phys. Soc.* **19**, 1153.

Phillips, O. M. 1970 On flows induced by diffusion in a stably stratified fluid. *Deep-Sea Res.* **17**, 435–443.

Turner, J. S. & Chen, C. F. 1974 Two-dimensional effects in double-diffusive convection. *J. Fluid Mech.* **63**, 577–592.

Wunsch, C. 1970 On oceanic boundary mixing. *Deep-Sea Res.* **17**, 293–301.